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# Quasi-invariance of Lebesgue measure under the homeomorphic flow generated by SDE with non-Lipschitz coefficient

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## Abstract

We consider the stochastic flow generated by Stratonovich stochastic differential equations with non-Lipschitz drift coefficients. Based on the author's previous works, we show that if the generalized divergence of the drift is bounded, then the Lebesgue measure on  $\mathbb{R}^d$  is quasi-invariant under the action of the stochastic flow, and the explicit expression of the Radon–Nikodym derivative is also presented. Finally we show in a special case that the unique solution of the corresponding Fokker–Planck equation is given by the density of the stochastic flow.

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**Keywords:** Stochastic flow; Stochastic differential equation; Quasi-invariance of measure; Non-Lipschitz condition; Fokker–Planck equation

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## 1. Introduction

In the past two decades, there is a trend to study the ordinary differential equations (abbreviated as ODE)

$$dX_{s,t} = A_0(X_{s,t}) dt, \quad t \geq s, \quad X_{s,s} = x \quad (1.1)$$

with irregular coefficients  $A_0: \mathbb{R}^d \rightarrow \mathbb{R}^d$  through analyzing the corresponding transport equation [4,5], or the continuity equation [1,2]. In the works [4,5], the authors developed the so-called

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DiPerna–Lions theory, which roughly speaking allows to obtain a flow of measurable maps determined by (1.1), after establishing the existence and uniqueness of solutions to the transport equation:

$$\frac{\partial u_{s,t}}{\partial t} + \langle A_0, \nabla u_{s,t} \rangle = 0, \quad t \geq s, \quad u_{s,s} = u_0.$$

More precisely, under the assumptions that the vector field  $A_0$  has a certain Sobolev regularity and its divergence is bounded or exponentially integrable, the authors proved that there exists a flow of measurable maps which solves the ODE and at the same time, leaves the reference measure quasi-invariant, that is, the push-forward of the reference measure is absolutely continuous with respect to itself. Different from this approach, Ambrosio [1,2] considered the continuity equation

$$\frac{\partial \mu_{s,t}}{\partial t} + D_x \cdot (A_0 \mu_{s,t}) = 0, \quad t \geq s, \quad \mu_{s,s} = \mu_0,$$

where  $D_x \cdot$  is the formal divergence, and proved that there exists a unique Lagrangian flow associated to the vector field  $A_0$ , provided that  $A_0$  has a Sobolev or BV regularity. This point of view was extended later on by Figalli [11] to study the relationship between the Itô SDE and the related Fokker–Planck (or forward Kolmogorov) equation. LeBris and Lions also considered this issue in a somewhat special situation (see [15, Section 5]), and they pointed out that the unique solution of the Fokker–Planck equation coincides with the density of the flow determined by the SDE. In the paper [17], the author studied the ODE (1.1) with log-Lipschitz coefficient, and showed that the push-forward of the Lebesgue measure by the solution has no charge on the subset whose Hausdorff dimension is less than that of the space, provided that the generalized divergence of the vector field  $A_0$  satisfies an integral condition. Here by the generalized divergence of  $A_0$ , we mean the locally integrable function  $\operatorname{div}(A_0)$  such that the following integration by parts formula holds:

$$\int_{\mathbb{R}^d} \phi \operatorname{div}(A_0) dx = - \int_{\mathbb{R}^d} \langle \nabla \phi, A_0 \rangle dx, \quad \text{for all } \phi \in C_c^\infty(\mathbb{R}^d),$$

where  $C_c^\infty(\mathbb{R}^d)$  is the totality of compactly supported smooth functions. In fact, a small modification of the proof in [17] shows that if  $\operatorname{div}(A_0)$  is locally bounded, then the Lebesgue measure is also quasi-invariant under the action of the solution  $X_{s,t}$ , see Section 2.

Recently motivated by the pioneer work of Malliavin [19], studying stochastic differential equations (SDE for short) with non-Lipschitz coefficients becomes one of the heat topics in stochastic analysis, cf. [6–10,17,18,22]. It is shown in [8] that if the drift coefficient  $A_0$  satisfies only the general Osgood condition (see Theorem 1 of [8] for details) and the diffusion coefficients belong to  $C_b^{3+\delta}$  with  $\delta > 0$ , then the solution  $X_{s,t}$  to the Stratonovich SDE

$$dX_{s,t} = \sum_{k=1}^N A_k(X_{s,t}) \circ dw_t^k + A_0(X_{s,t}) dt, \quad t \geq s, \quad X_{s,s} = x \quad (1.2)$$

is a stochastic flow of homeomorphisms, where  $w_t = (w_t^1, \dots, w_t^N)$  is a standard Brownian motion on  $\mathbb{R}^N$ . Though we use the same notation  $X_{s,t}$  to denote the solutions to Eqs. (1.1) and (1.2), there is no confusion according to the context. If in addition the vector field  $A_0$  satisfies a special log-Lipschitz condition and the generalized divergence  $\operatorname{div}(A_0)$  vanishes, then an explicit solution to the corresponding stochastic transport equation is also constructed there (Theorem 5.1

in [8]). A careful investigation of the proof indicates that the result also holds if  $\operatorname{div}(A_0)$  is locally square integrable, see Theorem 3.1 in Section 3. To state the main result of this work, we introduce some notations and hypotheses. We denote by  $\lambda$  the Lebesgue measure on  $\mathbb{R}^d$ . For some integer  $i \geq 1$ , define  $\log_{(i)}$  inductively by setting  $\log_{(1)} s = \log s$  and  $\log_{(i)} s = \log \log_{(i-1)} s$ . Let  $\rho_m(s) = (\log_{(1)} \frac{1}{s}) \cdots (\log_{(m)} \frac{1}{s})$ , where  $m \geq 1$  is an integer. Here are the conditions on  $A_0$ :

(H1) There exist  $m \geq 1$  and  $C > 0$ , such that for all  $|x - y| \leq c_0$ ,

$$|A_0(x) - A_0(y)| \leq C|x - y|\rho_m(|x - y|), \quad (1.3)$$

where  $c_0$  is small enough such that  $(0, c_0] \ni s \mapsto s\rho_m(s)$  is increasing and concave.

(H2) There exist  $C_0 > 0$  and  $\varepsilon_0 \in [0, 1)$  such that

$$|A_0(x)| \leq C_0(1 + |x|^{1-\varepsilon_0}), \quad \text{for all } x \in \mathbb{R}^d. \quad (1.4)$$

Similar to [4,5], we are able to prove the following theorem, under the help of the stochastic transport equation (3.1).

**Theorem 1.1.** *Suppose that the diffusion coefficients  $A_1, \dots, A_N$  belong to  $C_b^{3+\delta}$  for some  $\delta > 0$ , and the assumptions (H1) and (H2) hold with  $\varepsilon_0 \in (0, 1)$ . If the generalized divergence  $\operatorname{div}(A_0)$  of  $A_0$  is bounded, then the Lebesgue measure  $\lambda$  on  $\mathbb{R}^d$  is quasi-invariant under the flow  $X_{s,t}$  generated by (1.2). Moreover, the Radon–Nikodym derivative has the following explicit expression:*

$$\frac{d\lambda_{s,t}}{d\lambda}(x) = \exp\left(\sum_{k=1}^N \int_s^t \operatorname{div}(A_k)(X_{s,u}(x)) \circ dw_u^k + \int_s^t \operatorname{div}(A_0)(X_{s,u}(x)) du\right), \quad (1.5)$$

where  $\lambda_{s,t} = \lambda \circ X_{s,t}$ .

We would like to point out that we may take other measures  $\mu_0$  on  $\mathbb{R}^d$  as the reference measure, and it may also remain quasi-invariant under the action of the flows generated by (1.1) and (1.2). Indeed, when all the coefficients are smooth, Kunita proved (see Lemma 4.3.1 of [14]) a formula of the form (1.5), provided that  $\mu_0$  is absolutely continuous with respect to the Lebesgue measure  $\lambda$  and the corresponding Radon–Nikodym derivative  $\frac{d\mu_0}{d\lambda}$  belongs to  $C^3(\mathbb{R}^d)$  (a random version of the classical Liouville theorem). Based on this result, we can use the same limit procedure in this paper to deal with the general situation. However, we would not make this extension in this work, instead, we only treat the simpler case to illustrate the ideas.

The proof of Theorem 1.1 consists of many technical calculations, and here is a sketch. Theorem 1.1 is proved in two steps. For the first step, we show that the mean of the push-forward  $\lambda_{s,t}$  is dominated by a constant multiple of the Lebesgue measure, see Corollary 3.4. Then in the second step, we follow the method of Lemma 3.6 in [4] to prove that the Radon–Nikodym densities of smooth stochastic flows are convergent to (1.5) in a certain sense (Proposition 3.6), where the essential part is devoted to the term  $I_n^3$  involving  $\operatorname{div}(A_0)$ . We separate  $I_n^3$  into another two terms:  $I_n^{3,1}$  and  $I_n^{3,2}$ , where the term  $I_n^{3,1}$  is relatively easy to estimate. For the other term  $I_n^{3,2}$ , we deduce from the moment estimate (see Lemma 3.2) that most of the sample paths of the solution  $X_{s,t}(x)$  stay in a sufficiently big ball  $B(M)$ , on which the Lusin theorem is applied to get a measurable subset  $E_\varepsilon \subset B(M)$  such that the restriction  $\operatorname{div}(A_0)|_{E_\varepsilon}$  is uniformly continuous and  $\lambda(B(M) \setminus E_\varepsilon) < \varepsilon$ . Now according to whether  $X_{s,t}(x)$  and  $X_{s,t}^n(x)$  (the solution corresponding

to smooth drift  $A_0^n$ ) belong to  $E_\varepsilon$  or not, the result follows from a standard argument and the estimate obtained in the first step.

The paper is organized as follows. In Section 2, we first present a lemma that will be used in the latter sections, then we prove a result concerning the ODE (1.1), which improves the Main Theorem in [17]. Section 3 is the main part of this work, in which we consider the SDE (1.2), proving Theorem 1.1; after that we present a short discussion about the relationship between the solutions to the SDE and those of the related Fokker–Planck equation.

## 2. The case of ODE

The theories for ODE and SDE with smooth coefficients are well established (see for instance [13,14]), while the vector field  $A_0$  involved in this paper only satisfies the log-Lipschitz condition (1.3). To apply the known results, we approximate  $A_0$  by smooth vector fields and prove some limit theorems. Choose  $\chi \in C_c^\infty(\mathbb{R}^d, \mathbb{R}_+)$  such that  $\text{supp}(\chi) \subset B(1)$  and  $\int_{\mathbb{R}^d} \chi \, dx = 1$ , where  $B(R) = \{x \in \mathbb{R}^d : |x| \leq R\}$ . For  $n \geq 1$ , define  $\chi_n(x) = 2^{nd} \chi(2^n x)$ , then  $\chi_n \in C_c^\infty(\mathbb{R}^d, \mathbb{R}_+)$ ,  $\text{supp}(\chi_n) \subset B(2^{-n})$  and  $\int_{\mathbb{R}^d} \chi_n \, dx = 1$ . Set

$$A_0^n = A_0 * \chi_n,$$

where  $*$  is the convolution. First we present a lemma which will also be used in Section 3.

**Lemma 2.1.** Assume the conditions (H1) and (H2).

- (a) For all  $n \geq (\log \frac{1}{c_0}) / \log 2$ ,  $A_0^n \in C_b^\infty$ .
- (b) There exists  $C' > 0$  independent of  $n$ , such that  $|A_0^n(x)| \leq C'(1 + |x|^{1-\varepsilon_0})$ .
- (c) There is  $\zeta > 1$  such that  $\sup_{x \in \mathbb{R}^d} |A_0^n(x) - A_0(x)| \leq \zeta^{-n}$ .

**Proof.** (a) For  $i \in \{1, \dots, d\}$ , the  $i$ th component of the vector field  $A_0^n$  has the expression

$$A_0^{n,i}(x) = \int_{\mathbb{R}^d} A_0^i(y) \chi_n(x-y) \, dy = 2^{nd} \int_{\mathbb{R}^d} A_0^i(y) \chi(2^n(x-y)) \, dy.$$

For any multi-index  $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}_+^d$ ,

$$\begin{aligned} \frac{\partial^{|\alpha|} A_0^{n,i}}{\partial x^\alpha}(x) &= 2^{nd} \int_{\mathbb{R}^d} A_0^i(y) \frac{\partial^{|\alpha|} \chi}{\partial x^\alpha}(2^n(x-y)) \cdot (2^n)^{|\alpha|} \, dy \\ &= 2^{n|\alpha|} \int_{\mathbb{R}^d} A_0^i(x - 2^{-n}y) \frac{\partial^{|\alpha|} \chi}{\partial x^\alpha}(y) \, dy \\ &= 2^{n|\alpha|} \int_{\mathbb{R}^d} [A_0^i(x - 2^{-n}y) - A_0^i(x)] \frac{\partial^{|\alpha|} \chi}{\partial x^\alpha}(y) \, dy. \end{aligned}$$

Hence by (1.3), for  $n \geq (\log \frac{1}{c_0}) / \log 2$ ,

$$\left| \frac{\partial^{|\alpha|} A_0^{n,i}}{\partial x^\alpha}(x) \right| \leq 2^{n|\alpha|} \int_{B(1)} |A_0^i(x - 2^{-n}y) - A_0^i(x)| \cdot \left| \frac{\partial^{|\alpha|} \chi}{\partial x^\alpha}(y) \right| \, dy$$

$$\begin{aligned}
&\leq 2^{n|\alpha|} \int_{B(1)} C |2^{-n}y| \rho_m(|2^{-n}y|) \left| \frac{\partial^{|\alpha|} \chi}{\partial x^\alpha}(y) \right| dy \\
&\leq 2^{n(|\alpha|-1)} \rho_m(2^{-n}) C \int_{B(1)} \left| \frac{\partial^{|\alpha|} \chi}{\partial x^\alpha}(y) \right| dy,
\end{aligned}$$

where in the last step we have used the increasing property of  $s \mapsto s\rho_m(s)$  on  $(0, c_0]$ . (a) is proved. For the proof of (b) and (c), we refer the readers to Proposition 4.1 of [8].  $\square$

Let  $X_{s,t}^n(x)$  be the solution to (1.1) with  $A_0$  being replaced by  $A_0^n$ . Then by (c) of Lemma 2.1, it is easy to prove the following limit theorem (see [7]): for any  $T > s$  and  $R > 0$ ,

$$\lim_{n \rightarrow +\infty} \sup_{t \in [s, T]} \sup_{|x| \leq R} |X_{s,t}^n(x) - X_{s,t}(x)| = 0. \quad (2.1)$$

Now we are in the position to prove

**Theorem 2.2.** Assume the conditions (1.3) and (1.4) hold with  $\varepsilon_0 = 0$ . If in addition the generalized divergence  $\operatorname{div}(A_0)$  of the vector field  $A_0$  is locally bounded, then the Lebesgue measure  $\lambda$  is quasi-invariant under the flow generated by (1.1), i.e.  $\lambda_{s,t} := \lambda \circ X_{s,t} \ll \lambda$ , and the Radon–Nikodym derivative

$$K_{s,t}(x) = \frac{d\lambda_{s,t}}{d\lambda}(x) = \exp\left(\int_s^t \operatorname{div}(A_0)(X_{s,u}(x)) du\right).$$

**Proof.** Under these conditions, the solution of (1.1) defines a flow of homeomorphisms on  $\mathbb{R}^d$ , see [10, Section 2]. For  $\theta \in C_c^\infty(\mathbb{R}^d)$  and  $t \geq s$ , set  $\theta_{s,t}(x) = \theta(X_{s,t}^{-1}(x))$ . Then similar to (15) in [17], we have for any  $\psi \in C_c^\infty(\mathbb{R}^d)$ ,

$$\int_{\mathbb{R}^d} \theta_{s,t}(x) \psi(x) dx = \int_{\mathbb{R}^d} \theta(x) \psi(x) dx + \int_s^t \int_{\mathbb{R}^d} \theta_{s,u}(x) \operatorname{div}(\psi A_0)(x) dx du.$$

Proceed as in [17] and finally we arrive at

$$\begin{aligned}
\lambda_{s,t}(U) &= \lambda(U) + \int_s^t \int_U \operatorname{div}(A_0)(X_{s,u}(x)) d\lambda_{s,u}(x) du \\
&= \lambda(U) + \int_s^t \int_{X_{s,u}(U)} \operatorname{div}(A_0)(y) d\lambda du,
\end{aligned} \quad (2.2)$$

where  $U$  is any bounded open subset.

Fix any  $T > s$ . Let  $E$  be any measurable subset of  $\mathbb{R}^d$  such that  $\lambda(E) = 0$ . There is a partition  $\{E_n: n \geq 1\}$  of  $E$  with the property that  $\operatorname{diam}(E_n) \leq 1$  for all  $n \geq 1$ , where  $\operatorname{diam}(E_n)$  is the diameter of  $E_n$ . For any fixed  $n$ , there exists  $R > 0$  such that  $E_n \subset B(R)$ . Let  $R_T = (1+R)e^{TC_0}$ , then by (1.4) with  $\varepsilon_0 = 0$ ,  $\bigcup_{s \leq u \leq T} X_{s,u}(B(R)) \subset B(R_T)$ . Since  $\operatorname{div}(A_0)$  is locally bounded,

$$C_{T,R} := \sup_{y \in B(R_T)} |\operatorname{div}(A_0)(y)| < +\infty.$$

This and (2.2) imply that for any open subset  $U \subset B(R)$ ,  $\lambda_{s,t}(U)$  is Lipschitz continuous with respect to  $t \in [s, T]$ . From (2.2) we obtain

$$\begin{aligned}\lambda_{s,t}(U) &\leq \lambda(U) + \int_s^t \int_U |\operatorname{div}(A_0)(X_{s,u}(x))| d\lambda_{s,u}(x) du \\ &\leq \lambda(U) + C_{T,R} \int_s^t \lambda_{s,u}(U) du.\end{aligned}$$

Gronwall's inequality gives us  $\lambda_{s,t}(U) \leq \lambda(U)e^{(t-s)C_{T,R}}$ . Hence when restricted on the ball  $B(R)$ ,  $\lambda_{s,t}$  is absolutely continuous with respect to  $\lambda$ . This implies  $\lambda_{s,t}(E_n) = 0$ . Since  $n \geq 1$  is arbitrary, we get  $\lambda_{s,t}(E) = 0$ , which means  $\lambda_{s,t} \ll \lambda$ .

Now we prove that the Radon–Nikodym derivative  $\frac{d\lambda_{s,t}}{d\lambda}$  has the stated expression. Denote by  $\lambda_{s,t}^n = \lambda \circ X_{s,t}^n$ , then the classical Liouville theorem gives us

$$K_{s,t}^n(x) := \frac{d\lambda_{s,t}^n}{d\lambda}(x) = \exp\left(\int_s^t \operatorname{div}(A_0^n)(X_{s,u}^n(x)) du\right).$$

Since  $\operatorname{div}(A_0)$  is locally bounded, for all  $p \geq 1$ ,  $\operatorname{div}(A_0^n) = \operatorname{div}(A_0) * \chi_n$  converges in  $L_{loc}^p(\mathbb{R}^d)$  to  $\operatorname{div}(A_0)$ . For any  $R > 0$ ,

$$\begin{aligned}I_n &:= \int_{B(R)} \left| \int_s^t \operatorname{div}(A_0^n)(X_{s,u}^n(x)) du - \int_s^t \operatorname{div}(A_0)(X_{s,u}(x)) du \right| dx \\ &\leq \int_{B(R)} \int_s^t |\operatorname{div}(A_0^n)(X_{s,u}^n(x)) - \operatorname{div}(A_0)(X_{s,u}^n(x))| du dx \\ &\quad + \int_{B(R)} \int_s^t |\operatorname{div}(A_0)(X_{s,u}^n(x)) - \operatorname{div}(A_0)(X_{s,u}(x))| du dx \\ &=: I_n^1 + I_n^2.\end{aligned}$$

We have

$$\begin{aligned}I_n^1 &= \int_s^t \int_{B(R)} |\operatorname{div}(A_0^n)(X_{s,u}^n(x)) - \operatorname{div}(A_0)(X_{s,u}^n(x))| dx du \\ &= \int_s^t \int_{X_{s,u}^n(B(R))} |\operatorname{div}(A_0^n)(y) - \operatorname{div}(A_0)(y)| K_{s,u}^{n,-1}(y) dy du,\end{aligned}$$

where

$$K_{s,u}^{n,-1}(y) = \frac{d(\lambda \circ X_{s,u}^{n,-1})}{d\lambda}(y) = \exp\left(-\int_s^u \operatorname{div}(A_0^n)(X_{r,u}^{n,-1}(y)) dr\right).$$

Let  $R'_T = (1 + R)e^{TC'}$ , in which  $C'$  is the constant in Lemma 2.1(b). Recall that  $\varepsilon_0 = 0$ , by (b) of Lemma 2.1,

$$\bigcup_{n \geq 1} \bigcup_{u \in [s, T]} X_{s,u}^n(B(R)) \subset B(R'_T).$$

Since  $\operatorname{div}(A_0)$  is locally bounded, there is  $C'_{R,T} > 0$  independent of  $n$ , such that

$$\sup_{n \geq 1} \sup_{y \in B(R'_T)} |\operatorname{div}(A_0^n)(y)| \leq C'_{R,T}$$

for sufficiently large  $n$ , therefore  $K_{s,u}^{n,-1}(y) \leq e^{TC'_{R,T}}$  for all  $y \in X_{s,u}^n(B(R))$ ,  $u \in [s, T]$ . Hence we have

$$I_n^1 \leq T e^{TC'_{R,T}} \int_{B(R_T)} |\operatorname{div}(A_0^n)(y) - \operatorname{div}(A_0)(y)| dy \rightarrow 0$$

as  $n$  tends to  $+\infty$ . By (2.1) and following the method of Lemma 3.6 in [4], we can prove that  $\operatorname{div}(A_0)(X_{s,u}^n(x))$  converges to  $\operatorname{div}(A_0)(X_{s,u}(x))$  in the Lebesgue measure on  $[s, T] \times B(R)$ . Hence the dominated convergence theorem leads to

$$I_n^2 = \int_s^t \int_{B(R)} |\operatorname{div}(A_0)(X_{s,u}^n(x)) - \operatorname{div}(A_0)(X_{s,u}(x))| dx du \rightarrow 0$$

as  $n \rightarrow +\infty$ . In one word, we have proved that  $I_n \rightarrow 0$ . Therefore up to a subsequence, for a.e.  $x \in \mathbb{R}^d$ ,

$$\int_s^t \operatorname{div}(A_0^n)(X_{s,u}^n(x)) du \rightarrow \int_s^t \operatorname{div}(A_0)(X_{s,u}(x)) du.$$

This implies that  $K_{s,t}^n$  converges to  $K_{s,t}$  for  $\lambda$ -a.e.  $x \in \mathbb{R}^d$ . Now fix an arbitrary  $f \in C_c(\mathbb{R}^d)$ . By what we have just proved and the dominated convergence theorem,

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^d} f(x) K_{s,t}^n(x) dx = \int_{\mathbb{R}^d} f(x) K_{s,t}(x) dx.$$

On the other hand,

$$\int_{\mathbb{R}^d} f(x) K_{s,t}^n(x) dx = \int_{\mathbb{R}^d} f(x) d\lambda_{s,t}^n(x) = \int_{\mathbb{R}^d} f(X_{s,t}^{n,-1}(y)) dy$$

which converges to

$$\int_{\mathbb{R}^d} f(X_{s,t}^{-1}(y)) dy = \int_{\mathbb{R}^d} f(x) d\lambda_{s,t}(x)$$

by (2.1). Therefore

$$\int_{\mathbb{R}^d} f(x) K_{s,t}(x) dx = \int_{\mathbb{R}^d} f(x) d\lambda_{s,t}(x).$$

The proof is completed.  $\square$

**Remark 2.3.** The expression of the Radon–Nikodym derivative can also be deduced heuristically in the following way. By (2.2) we have

$$\begin{aligned} \int_U K_{s,t}(x) d\lambda &= \int_U 1 d\lambda + \int_s^t \int_U \operatorname{div}(A_0)(X_{s,u}(x)) K_{s,u}(x) d\lambda du \\ &= \int_U \left( 1 + \int_s^t \operatorname{div}(A_0)(X_{s,u}(x)) K_{s,u}(x) du \right) d\lambda. \end{aligned}$$

This equality holds for any bounded open subset  $U$ , therefore

$$K_{s,t}(x) = 1 + \int_s^t \operatorname{div}(A_0)(X_{s,u}(x)) K_{s,u}(x) du,$$

which leads to the desired result.  $\square$

### 3. The case of SDE

In this section, we consider the Stratonovich SDE (1.2) with smooth diffusion coefficients  $A_k$ , and irregular drift coefficient  $A_0$  satisfying the conditions (1.3) and (1.4) with  $\varepsilon_0 > 0$ . We prove Theorem 1.1 in Section 3.1, which involves many technical calculations and constitutes the main part of this paper. After that in Section 3.2, we give a sufficient condition for the uniqueness of the relating Fokker–Planck equation. It is also shown that the unique solution has an explicit density with respect to the Lebesgue measure  $\lambda$ , provided that the initial datum is absolutely continuous with respect to  $\lambda$ .

#### 3.1. Quasi-invariance of Lebesgue measure

We first generalize Theorem 5.1 in [8] to the case that  $\operatorname{div}(A_0)$  is locally square integrable.

**Theorem 3.1.** Assume that the diffusion coefficients  $A_1, \dots, A_N$  belong to  $C_b^{3+\delta}$  for some  $\delta > 0$ , and the conditions (H1) and (H2) hold with  $\varepsilon_0 \in (0, 1)$ . Suppose that the generalized divergence  $\operatorname{div}(A_0) \in L^2_{loc}(\mathbb{R}^d)$ . Let  $\theta_0$  be a continuous function on  $\mathbb{R}^d$ , having polynomial growth:

$$|\theta_0(x)| \leq C(1 + |x|^{q_0}), \quad x \in \mathbb{R}^d.$$

Then  $\theta_{s,t}(x) := \theta_0(X_{s,t}^{-1}(x))$  solves the stochastic transport equation in distributional sense, that is, for any  $\phi \in C_c^\infty(\mathbb{R}^d)$ ,

$$(\theta_{s,t}, \phi)_{L^2} = (\theta_0, \phi)_{L^2} + \sum_{k=1}^N \int_s^t (\theta_{s,u}, \operatorname{div}(\phi A_k))_{L^2} \circ dw_u^k + \int_s^t (\theta_{s,u}, \operatorname{div}(\phi A_0))_{L^2} du, \quad (3.1)$$

where  $(\cdot, \cdot)_{L^2}$  denotes the inner product in  $L^2(\mathbb{R}^d, \lambda)$ .



**Proof.** The proof is in general similar to that of Theorem 5.1 of [8]. We only have to modify the steps that involve the term  $\operatorname{div}(A_0)$ . The last sentence on p. 1102 and the first sentence on p. 1103 of [8] should be replaced by the arguments below. We have

$$\operatorname{div}(\phi A_0^n) = \langle \nabla \phi, A_0^n \rangle + \phi \operatorname{div}(A_0^n).$$

By (c) of Lemma 2.1, the first term on the right side tends to  $\langle \nabla \phi, A_0 \rangle$  uniformly on  $\mathbb{R}^d$ , hence

$$\mathbb{E} \left( \left| \int_s^t (\theta_{s,u}^n, \langle \nabla \phi, A_0^n \rangle)_{L^2} du - \int_s^t (\theta_{s,u}, \langle \nabla \phi, A_0 \rangle)_{L^2} du \right|^2 \right) \rightarrow 0 \quad \text{as } n \rightarrow +\infty, \quad (\text{i})$$

where  $\theta_{s,u}^n(x) = \theta_0((X_{s,u}^n)^{-1}(x))$ , and  $X_{s,t}^n(x)$  is the solution of (1.2) with  $A_0$  being replaced by  $A_0^n$ . Furthermore, since

$$\operatorname{div}(A_0^n) = \operatorname{div}(A_0 * \chi_n) = \operatorname{div}(A_0) * \chi_n$$

and  $\operatorname{div}(A_0)$  is locally square integrable,  $\phi \operatorname{div}(A_0^n)$  converges to  $\phi \operatorname{div}(A_0)$  in  $L^2(\mathbb{R}^d, \lambda)$ . It follows that when  $n \rightarrow +\infty$ ,

$$\mathbb{E} \left( \left| \int_s^t (\theta_{s,u}^n, \phi \operatorname{div}(A_0^n))_{L^2} du - \int_s^t (\theta_{s,u}, \phi \operatorname{div}(A_0))_{L^2} du \right|^2 \right) \rightarrow 0. \quad (\text{ii})$$

From (i) and (ii), we conclude that the third term on the right side of (5.10) in [8] converges to  $\int_s^t (\theta_{s,u}, \operatorname{div}(\phi A_0))_{L^2} du$  in  $L^2(\Omega)$ . Since the rest of the proof are the same as those of Theorem 5.1 in [8], we omit them.  $\square$

Fix any  $s \in [0, 1)$ . Next we give a result concerning the spatial growth of the solution  $X_{s,t}(x)$  to (1.2).

**Lemma 3.2.** *There exist  $F \in \bigcap_{p>1} L^p(\Omega)$  and  $\beta > 1$ , such that*

$$|X_{s,t}(x)| \leq F \cdot (1 + |x|^\beta), \quad \text{for all } (t, x) \in [s, 1] \times \mathbb{R}^d. \quad (3.2)$$

**Proof.** It follows from [20] (see also (2.7) of [8]) that the solution  $X_{s,t}(x)$  to (1.2) can be represented as

$$X_{s,t}(x) = \varphi_{s,t}(Y_{s,t}(x)), \quad (\text{i})$$

where  $\varphi_{s,t}(x)$  is the solution to the Stratonovich SDE without drift:

$$d\varphi_{s,t} = \sum_{k=1}^N A_k(\varphi_{s,t}) \circ dw_t^k, \quad t \geq s, \quad \varphi_{s,s} = x,$$

and  $Y_{s,t}(x)$  solves the ODE below:

$$dY_{s,t} = \tilde{A}_0(s, t, Y_{s,t}) dt, \quad t \geq s, \quad Y_{s,s} = x$$

with  $\tilde{A}_0(s, t, x) = (J_{s,t}(x))^{-1} A_0(\varphi_{s,t}(x))$ , where  $J_{s,t}(x)$  is the Jacobian matrix  $\partial_x \varphi_{s,t}(x)$ . It is known (see [3] or (2.9) of [8]) that there exist  $F_0 \in \bigcap_{p>1} L^p(\Omega)$  and  $\beta > 1$ , such that for all  $(t, x) \in [s, 1] \times \mathbb{R}^d$ ,

$$|\varphi_{s,t}(x)| \leq F_0 \cdot (1 + |x|^\beta). \quad (\text{ii})$$

Moreover, by the definition of  $Y_{s,t}(x)$  and Lemma 2.2 of [8], we have

$$\begin{aligned} |Y_{s,t}(x)| &\leq |x| + \int_s^t \Phi_1 \cdot (1 + |Y_{s,u}(x)|^{1-\varepsilon_1}) du \\ &\leq |x| + \Phi_1 + \int_s^t (\varepsilon_1 \Phi_1^{1/\varepsilon_1} + (1 - \varepsilon_1) |Y_{s,u}(x)|) du \\ &\leq |x| + \Phi_1 + \varepsilon_1 \Phi_1^{1/\varepsilon_1} + (1 - \varepsilon_1) \int_s^t |Y_{s,u}(x)| du. \end{aligned}$$

The Gronwall inequality gives us

$$|Y_{s,t}(x)| \leq (|x| + \Phi_1 + \varepsilon_1 \Phi_1^{1/\varepsilon_1}) e^{1-\varepsilon_1} \leq \tilde{\Phi} \cdot (1 + |x|), \quad (\text{iii})$$

where  $\tilde{\Phi} = e(1 + \Phi_1 + \varepsilon_1 \Phi_1^{1/\varepsilon_1})$ . It is clear that  $\tilde{\Phi}$  belongs to all  $L^p(\Omega)$ . Now the desired result follows easily from (i)–(iii).  $\square$

By (3.2), for any  $R > 0$ ,  $X_{s,t}(B(R)) \subset B(F(1 + R^\beta))$ . It follows that

$$\lambda_{s,t}(B(R)) = \lambda(X_{s,t}(B(R))) \leq \lambda(B(F(1 + R^\beta))) = c_d F^d (1 + R^\beta)^d,$$

where  $c_d$  is the volume of the unit ball in  $\mathbb{R}^d$ . Hence for all  $p \geq 1$ ,

$$\mathbb{E} \left( \sup_{t \in [s, 1]} \lambda_{s,t}^p(B(R)) \right) \leq c_d^p (1 + R^\beta)^{dp} \mathbb{E}(F^{dp}) < +\infty. \quad (3.3)$$

Now under the help of the stochastic transport equation (3.1), we are able to obtain some estimates on  $\lambda_{s,t} = \lambda \circ X_{s,t}$ .

**Proposition 3.3.** *Under the assumptions of Theorem 1.1, for any  $p \geq 1$ , there exists  $C_p > 0$  such that for all bounded domain  $U \subset \mathbb{R}^d$ , we have*

$$\mathbb{E} \left( \sup_{t \in [s, 1]} \lambda_{s,t}^p(U) \right) \leq C_p \lambda^p(U).$$

**Proof.** We first fix a continuous function  $\theta_0$  with compact support. There is  $R > 0$  such that  $\text{supp}(\theta_0) \subset B(R)$ . Let  $\Phi \in C_c^\infty(\mathbb{R}^d, [0, 1])$  satisfy  $\Phi(x) = 1$  for  $|x| \leq 1$ , and  $\Phi(x) = 0$  for  $|x| \geq 2$ . For  $n \geq 1$ , set  $\Phi_n(x) = \Phi(\frac{x}{n})$ . Replacing  $\phi$  by  $\Phi_n$  in (3.1) and passing to the Itô integral, we get

$$\begin{aligned} (\theta_{s,t}, \Phi_n)_{L^2} &= (\theta_0, \Phi_n)_{L^2} + \sum_{k=1}^N \int_s^t (\theta_{s,u}, \text{div}(\Phi_n A_k))_{L^2} dw_u^k + \int_s^t (\theta_{s,u}, \text{div}(\Phi_n A_0))_{L^2} du \\ &\quad + \frac{1}{2} \sum_{k=1}^N \int_s^t (\theta_{s,u}, \text{div}(A_k \text{div}(\Phi_n A_k)))_{L^2} du. \end{aligned} \quad (3.4)$$

In the following we prove that both sides of the above equality converge respectively as  $n \rightarrow +\infty$ . We have

$$|(\theta_{s,t}, \Phi_n)_{L^2} - (\theta_{s,t}, 1)_{L^2}| \leq \int_{\mathbb{R}^d} |\theta_{s,t}(x)| d\lambda = \int_{\mathbb{R}^d} |\theta_0(y)| d\lambda_{s,t}(y) \leq \|\theta_0\|_\infty \lambda_{s,t}(B(R)).$$

Thus by (3.3),

$$\mathbb{E} \left( \sup_{t \in [s, 1]} |(\theta_{s,t}, \Phi_n)_{L^2} - (\theta_{s,t}, 1)_{L^2}|^2 \right) \leq \|\theta_0\|_\infty^2 \mathbb{E} \left( \sup_{t \in [s, 1]} \lambda_{s,t}^2(B(R)) \right) < +\infty.$$

Moreover, for a.s.  $w \in \Omega$ ,  $\bigcup_{t \in [s, 1]} X_{s,t}(\text{supp}(\theta_0)) \subset B(F(1 + R^\beta))$  is a bounded subset of  $\mathbb{R}^d$ . Consequently, if  $n$  is big enough,

$$(\theta_{s,t}, \Phi_n)_{L^2} - (\theta_{s,t}, 1)_{L^2} = \int_{X_{s,t}(\text{supp}(\theta_0))} \theta_{s,t}(x) (\Phi_n(x) - 1) d\lambda = 0 \quad \text{for all } t \in [s, 1].$$

That is to say, a.s.  $\sup_{t \in [s, 1]} |(\theta_{s,t}, \Phi_n)_{L^2} - (\theta_{s,t}, 1)_{L^2}|$  converges to 0 as  $n \rightarrow +\infty$ . Therefore by the dominated convergence theorem, we obtain

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left( \sup_{t \in [s, 1]} |(\theta_{s,t}, \Phi_n)_{L^2} - (\theta_{s,t}, 1)_{L^2}|^2 \right) = 0.$$

Next, by the B rkh lder inequality,

$$\begin{aligned} & \mathbb{E} \left( \left| \sum_{k=1}^N \int_s^t (\theta_{s,u}, \text{div}(\Phi_n A_k))_{L^2} dw_u^k - \sum_{k=1}^N \int_s^t (\theta_{s,u}, \text{div}(A_k))_{L^2} dw_u^k \right|^2 \right) \\ & \leq \mathbb{E} \left( \sum_{k=1}^N \int_s^1 (\theta_{s,u}, \text{div}(\Phi_n A_k) - \text{div}(A_k))_{L^2}^2 du \right). \end{aligned}$$

Since  $\text{div}(\Phi_n A_k) = \langle \nabla \Phi_n, A_k \rangle + \Phi_n \text{div}(A_k)$  and

$$\begin{aligned} |\langle \nabla \Phi_n, A_k \rangle(x)| & \leq |\nabla \Phi_n(x)| \cdot |A_k(x)| \leq \frac{1}{n} \|\nabla \Phi\|_\infty \mathbf{1}_{\{|n| \leq |x| \leq 2n\}}(x) \cdot C(1 + |x|) \\ & \leq C \|\nabla \Phi\|_\infty \frac{1 + 2n}{n} < +\infty, \end{aligned}$$

$\text{div}(\Phi_n A_k)$  are uniformly bounded ( $1 \leq k \leq N$ ,  $n \geq 1$ ), therefore we have

$$\sum_{k=1}^N \int_s^1 (\theta_{s,u}, \text{div}(\Phi_n A_k) - \text{div}(A_k))_{L^2}^2 du \leq C' \|\theta_0\|_\infty^2 \sup_{u \in [s, 1]} \lambda_{s,u}^2(B(R)),$$

whose right-hand side is integrable with respect to  $\mathbb{P}$ . By the same reasoning as above, we know that a.s., the left side of the above inequality tends to 0 as  $n \rightarrow +\infty$ . Therefore applying the dominated convergence theorem once more, we get

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left( \left| \sum_{k=1}^N \int_s^t (\theta_{s,u}, \text{div}(\Phi_n A_k))_{L^2} dw_u^k - \sum_{k=1}^N \int_s^t (\theta_{s,u}, \text{div}(A_k))_{L^2} dw_u^k \right|^2 \right) = 0.$$

In the same way, we can prove the last two terms of (3.4) converge in  $L^2(\Omega)$  respectively. Thus we have by letting  $n \rightarrow +\infty$  on both sides of (3.4),

$$\begin{aligned} (\theta_{s,t}, 1)_{L^2} &= (\theta_0, 1)_{L^2} + \sum_{k=1}^N \int_s^t (\theta_{s,u}, \operatorname{div}(A_k))_{L^2} dw_u^k + \int_s^t (\theta_{s,u}, \operatorname{div}(A_0))_{L^2} du \\ &\quad + \frac{1}{2} \sum_{k=1}^N \int_s^t (\theta_{s,u}, \operatorname{div}(A_k \operatorname{div}(A_k)))_{L^2} du. \end{aligned}$$

This is equivalent to

$$\begin{aligned} \int_{\mathbb{R}^d} \theta_0(y) d\lambda_{s,t}(y) &= \int_{\mathbb{R}^d} \theta_0(y) d\lambda(y) + \sum_{k=1}^N \int_s^t \int_{\mathbb{R}^d} \theta_0(y) \operatorname{div}(A_k)(X_{s,u}(y)) d\lambda_{s,u}(y) dw_u^k \\ &\quad + \int_s^t \int_{\mathbb{R}^d} \theta_0(y) \operatorname{div}(A_0)(X_{s,u}(y)) d\lambda_{s,u}(y) du \\ &\quad + \frac{1}{2} \sum_{k=1}^N \int_s^t \int_{\mathbb{R}^d} \theta_0(y) \operatorname{div}(A_k \operatorname{div}(A_k))(X_{s,u}(y)) d\lambda_{s,u}(y) du. \end{aligned} \quad (3.5)$$

Let  $U \subset \mathbb{R}^d$  be a bounded domain. Then there is a sequence of functions  $\{\theta_n: n \geq 1\} \subset C_c(\mathbb{R}^d)$ , such that  $\theta_n(y) \uparrow \mathbf{1}_U(y)$ , for all  $y \in \mathbb{R}^d$ . Replace  $\theta_0$  by  $\theta_n$  in (3.5) and let  $n \rightarrow +\infty$ , we have by repeating the procedure of proving (3.5),

$$\begin{aligned} \lambda_{s,t}(U) &= \lambda(U) + \sum_{k=1}^N \int_s^t \int_U \operatorname{div}(A_k)(X_{s,u}(y)) d\lambda_{s,u}(y) dw_u^k \\ &\quad + \int_s^t \int_U \operatorname{div}(A_0)(X_{s,u}(y)) d\lambda_{s,u}(y) du \\ &\quad + \frac{1}{2} \sum_{k=1}^N \int_s^t \int_U \operatorname{div}(A_k \operatorname{div}(A_k))(X_{s,u}(y)) d\lambda_{s,u}(y) du, \quad \text{for all } t \in [s, 1]. \end{aligned} \quad (3.6)$$

It suffices to prove the estimate for  $p > 2$ . There exists  $C_p > 0$  such that

$$\begin{aligned} \lambda_{s,t}^p(U) &\leq C_p \lambda^p(U) + C_p \left| \sum_{k=1}^N \int_s^t \int_U \operatorname{div}(A_k)(X_{s,u}(y)) d\lambda_{s,u}(y) dw_u^k \right|^p \\ &\quad + C_p \left| \int_s^t \int_U \operatorname{div}(A_0)(X_{s,u}(y)) d\lambda_{s,u}(y) du \right|^p \\ &\quad + C_p \left| \frac{1}{2} \sum_{k=1}^N \int_s^t \int_U \operatorname{div}(A_k \operatorname{div}(A_k))(X_{s,u}(y)) d\lambda_{s,u}(y) du \right|^p. \end{aligned}$$

For any  $T \in [s, 1]$ , the B urk older inequality gives us

$$\begin{aligned} & \mathbb{E} \left( \sup_{s \leq t \leq T} \left| \sum_{k=1}^N \int_s^t \int_U \operatorname{div}(A_k)(X_{s,u}(y)) d\lambda_{s,u}(y) dw_u^k \right|^p \right) \\ & \leq C'_p \mathbb{E} \left( \sum_{k=1}^N \int_s^T \left( \int_U \operatorname{div}(A_k)(X_{s,u}(y)) d\lambda_{s,u}(y) \right)^2 du \right)^{\frac{p}{2}} \\ & \leq C'_p \mathbb{E} \left( \sum_{k=1}^N \|\operatorname{div}(A_k)\|_\infty^2 \int_s^T \lambda_{s,u}^2(U) du \right)^{\frac{p}{2}} \leq C''_p \int_s^T \mathbb{E}(\lambda_{s,u}^p(U)) du, \end{aligned}$$

where  $C''_p$  is dependent on  $\sum_{k=1}^N \|\operatorname{div}(A_k)\|_\infty^2$ . Similarly, since  $\operatorname{div}(A_0)$  and  $\operatorname{div}(A_k \operatorname{div}(A_k))$  are bounded, by the Cauchy inequality, we finally get

$$\mathbb{E} \left( \sup_{s \leq t \leq T} \lambda_{s,t}^p(U) \right) \leq C_{p,1} \lambda^p(U) + C_{p,2} \int_s^T \mathbb{E} \left( \sup_{s \leq u \leq t} \lambda_{s,u}^p(U) \right) dt.$$

The Gronwall inequality leads to

$$\mathbb{E} \left( \sup_{s \leq t \leq T} \lambda_{s,t}^p(U) \right) \leq C_{p,1} \lambda^p(U) e^{C_{p,2}}, \quad \text{for all } T \in [s, 1].$$

Therefore we have finished proving the result.  $\square$

We can deduce the following useful results from Proposition 3.3.

#### Corollary 3.4.

(1) For any measurable subset  $E \subset \mathbb{R}^d$ ,

$$\mathbb{E} \left( \sup_{t \in [s, 1]} \lambda_{s,t}(E) \right) \leq C_1 \lambda(E),$$

where  $C_1 > 0$  is the constant in Proposition 3.3. The same result also holds for  $\lambda_{s,t}^{-1} := \lambda \circ X_{s,t}^{-1}$ .

(2) If  $E \subset \mathbb{R}^d$  is  $\lambda$ -negligible, then a.s. for all  $t \in [s, 1]$ ,  $\lambda_{s,t}(E) = 0$ .

**Proof.** (1) It suffices to prove the result for the case  $\lambda(E) < +\infty$ . By the measure theory, for any  $\varepsilon > 0$ , there exists a sequence of balls  $\{U_n: n \geq 1\}$ , satisfying  $E \subset \bigcup_{n \geq 1} U_n$  and  $\sum_{n=1}^{+\infty} \lambda(U_n) < \lambda(E) + \varepsilon$ . Applying Proposition 3.3 with  $p = 1$ , we have

$$\mathbb{E} \left( \sup_{t \in [s, 1]} \lambda_{s,t}(E) \right) \leq \sum_{n=1}^{+\infty} \mathbb{E} \left( \sup_{t \in [s, 1]} \lambda_{s,t}(U_n) \right) \leq \sum_{n=1}^{+\infty} C_1 \lambda(U_n) \leq C_1 (\lambda(E) + \varepsilon).$$

Since  $\varepsilon > 0$  is arbitrary, we conclude the result.

(2) If  $E \subset \mathbb{R}^d$  is  $\lambda$ -negligible, then by (1), we have  $\mathbb{E}(\sup_{t \in [s, 1]} \lambda_{s,t}(E)) = 0$ , from which the assertion follows.  $\square$

Notice that Corollary 3.4 is not enough to conclude Theorem 1.1, since the exceptional set is dependent on the negligible subset  $E \subset \mathbb{R}^d$ . To prove Theorem 1.1, we need some more preparations.

For  $n \geq 1$ , let  $X_{s,t}^n(x)$  be the solution to (1.2) with  $A_0$  being replaced by  $A_0^n$ . Then by Theorem 4.2 of [8], the following limit theorem holds: for any  $p > 1$  and  $R > 0$ ,

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left( \sup_{t \in [s,1]} \sup_{|x| \leq R} |X_{s,t}^n(x) - X_{s,t}(x)|^p \right) = 0. \quad (3.7)$$

Let  $\lambda_{s,t}^n = \lambda \circ X_{s,t}^n$  and

$$K_{s,t}^n(x) = \exp \left( \sum_{i=1}^N \int_s^t \operatorname{div}(A_i)(X_{s,u}^n(x)) \circ dw_u^i + \int_s^t \operatorname{div}(A_0^n)(X_{s,u}^n(x)) du \right).$$

Then Lemma 4.3.1 of [14] tells us that  $\lambda_{s,t}^n \ll \lambda$  and

$$K_{s,t}^n(x) = \frac{d\lambda_{s,t}^n}{d\lambda}(x). \quad (3.8)$$

Moreover the sequence  $K_{s,t}^n(x)$  is bounded in all  $L^p$ .

**Lemma 3.5.** Assume the conditions of Theorem 1.1. Then for any  $p > 1$ ,

$$\sup_{n \geq 1} \sup_{t \in [s,1]} \sup_{x \in \mathbb{R}^d} \mathbb{E}[(K_{s,t}^n(x))^p] < +\infty.$$

**Proof.** To simplify the notations, denote by  $\xi_i = \operatorname{div}(A_i)$ ,  $\eta_i = \langle \nabla(\operatorname{div}(A_i)), A_i \rangle$ ,  $i = 1, \dots, N$ , and  $\xi_0^n = \operatorname{div}(A_0^n)$ . It is clear that  $\xi_i$ ,  $\eta_i$ ,  $i = 1, \dots, N$ , are bounded. According to the definition of  $A_0^n$ ,

$$\xi_0^n = \operatorname{div}(A_0^n) = \operatorname{div}(A_0) * \chi_n,$$

hence it is uniformly bounded in  $n \geq 1$ .

By Eq. (1.2) with  $A_0$  being replaced by  $A_0^n$ , for all  $s \leq t \leq 1$ ,

$$\begin{aligned} (K_{s,t}^n(x))^p &= \exp \left( p \sum_{i=1}^N \int_s^t \xi_i(X_{s,u}^n(x)) \circ dw_u^i + p \int_s^t \xi_0^n(X_{s,u}^n(x)) du \right) \\ &= \exp \left( p \sum_{i=1}^N \int_s^t \xi_i(X_{s,u}^n(x)) dw_u^i + \frac{p}{2} \sum_{i=1}^N \int_s^t \eta_i(X_{s,u}^n(x)) du \right. \\ &\quad \left. + p \int_s^t \xi_0^n(X_{s,u}^n(x)) du \right) \\ &= \exp \left( p \sum_{i=1}^N \int_s^t \xi_i(X_{s,u}^n(x)) dw_u^i - \frac{p^2}{2} \sum_{i=1}^N \int_s^t (\xi_i(X_{s,u}^n(x)))^2 du \right) \\ &\quad \times \exp \left( \frac{p}{2} \sum_{i=1}^N \int_s^t [p(\xi_i(X_{s,u}^n(x)))^2 + \eta_i(X_{s,u}^n(x))] du \right) \end{aligned}$$

$$\begin{aligned}
& + p \int_s^t \xi_0^n(X_{s,u}^n(x)) du \Big) \\
& \leq \exp \left( p \sum_{i=1}^N \int_s^t \xi_i(X_{s,u}^n(x)) dw_u^i - \frac{p^2}{2} \sum_{i=1}^N \int_s^t (\xi_i(X_{s,u}^n(x)))^2 du \right) \\
& \quad \times \exp \left( \frac{p}{2} \sum_{i=1}^N (p \|\xi_i\|_\infty^2 + \|\eta_i\|_\infty) + p \|\xi_0^n\|_\infty \right).
\end{aligned}$$

By the Novikov theorem (see [13, p. 152, Theorem 5.3]) and the boundedness of  $\xi_i$ ,  $i = 1, \dots, N$ , it is easy to know that the first term on the right side is an exponential martingale, hence

$$\mathbb{E}((K_{s,t}^n(x))^p) \leq \exp \left( \frac{p}{2} \sum_{i=1}^N [p \|\xi_i\|_\infty^2 + \|\eta_i\|_\infty] + p \|\xi_0^n\|_\infty \right)$$

whose right side is dominated by a constant independent of  $n$ ,  $t$  and  $x$ .  $\square$

We introduce two new notations. Set

$$\zeta_{s,t}^n(x) = \sum_{i=1}^N \int_s^t \operatorname{div}(A_i)(X_{s,u}^n(x)) \circ dw_u^i + \int_s^t \operatorname{div}(A_0^n)(X_{s,u}^n(x)) du$$

and

$$\zeta_{s,t}(x) = \sum_{i=1}^N \int_s^t \operatorname{div}(A_i)(X_{s,u}(x)) \circ dw_u^i + \int_s^t \operatorname{div}(A_0)(X_{s,u}(x)) du.$$

We have the following key result.

**Proposition 3.6.** Assume the conditions of Theorem 1.1, then up to a subsequence, for any  $p \geq 2$  and  $R > 0$ ,

$$\lim_{n \rightarrow +\infty} \int_{B(R)} \mathbb{E}(|\zeta_{s,t}^n(x) - \zeta_{s,t}(x)|^p) dx = 0.$$

**Proof.** Recall that  $\xi_i = \operatorname{div}(A_i)$ ,  $\eta_i = \langle \nabla(\operatorname{div}(A_i)), A_i \rangle$ ,  $i = 1, \dots, N$ , and  $\xi_0^n = \operatorname{div}(A_0^n)$ . We also denote by  $\xi_0 = \operatorname{div}(A_0)$ . By Eq. (1.2),

$$\zeta_{s,t}(x) = \sum_{i=1}^N \int_s^t \xi_i(X_{s,u}(x)) dw_u^i + \frac{1}{2} \sum_{i=1}^N \int_s^t \eta_i(X_{s,u}(x)) du + \int_s^t \xi_0(X_{s,u}(x)) du.$$

Replacing  $A_0$  by  $A_0^n$  in (1.2), we also have

$$\zeta_{s,t}^n(x) = \sum_{i=1}^N \int_s^t \xi_i(X_{s,u}^n(x)) dw_u^i + \frac{1}{2} \sum_{i=1}^N \int_s^t \eta_i(X_{s,u}^n(x)) du + \int_s^t \xi_0^n(X_{s,u}^n(x)) du.$$

Thus

$$\begin{aligned}
 |\zeta_{s,t}^n(x) - \zeta_{s,t}(x)|^p &\leq C_p \left| \sum_{i=1}^N \int_s^t \xi_i(X_{s,u}^n(x)) dw_u^i - \sum_{i=1}^N \int_s^t \xi_i(X_{s,u}(x)) dw_u^i \right|^p \\
 &\quad + C_p \left| \frac{1}{2} \sum_{i=1}^N \int_s^t \eta_i(X_{s,u}^n(x)) du - \frac{1}{2} \sum_{i=1}^N \int_s^t \eta_i(X_{s,u}(x)) du \right|^p \\
 &\quad + C_p \left| \int_s^t \xi_0^n(X_{s,u}^n(x)) du - \int_s^t \xi_0(X_{s,u}(x)) du \right|^p \\
 &=: C_p (I_n^1 + I_n^2 + I_n^3).
 \end{aligned}$$

By the Burkholder inequality and the boundedness of  $\nabla \xi_i$ , we have

$$\begin{aligned}
 \mathbb{E}(I_n^1) &\leq C_p' \mathbb{E} \left( \sum_{i=1}^N \int_s^t (\xi_i(X_{s,u}^n(x)) - \xi_i(X_{s,u}(x)))^2 du \right)^{\frac{p}{2}} \\
 &\leq C_p'' \left( \sum_{i=1}^N \|\nabla \xi_i\|_\infty^2 \right)^{\frac{p}{2}} \int_s^t \mathbb{E}(|X_{s,u}^n(x) - X_{s,u}(x)|^p) du.
 \end{aligned}$$

Consequently, by (3.7),

$$\int_{B(R)} \mathbb{E}(I_n^1) dx \leq \tilde{C}_{p,R} \mathbb{E} \left( \sup_{u \in [s,1]} \sup_{|x| \leq R} |X_{s,u}^n(x) - X_{s,u}(x)|^p \right) \rightarrow 0$$

as  $n \rightarrow +\infty$ . In the same way we have

$$\lim_{n \rightarrow +\infty} \int_{B(R)} \mathbb{E}(I_n^2) dx = 0.$$

Now let us deal with the most difficult term  $I_n^3$ .

$$\begin{aligned}
 I_n^3 &\leq C_p \int_s^t |\xi_0^n(X_{s,u}^n(x)) - \xi_0(X_{s,u}^n(x))|^p du + C_p \int_s^t |\xi_0(X_{s,u}^n(x)) - \xi_0(X_{s,u}(x))|^p du \\
 &=: C_p (I_n^{3,1} + I_n^{3,2}).
 \end{aligned}$$

We have by the Fubini theorem,

$$\begin{aligned}
 \int_{B(R)} \mathbb{E}(I_n^{3,1}) dx &= \int_s^t \mathbb{E} \left( \int_{B(R)} |\xi_0^n(X_{s,u}^n(x)) - \xi_0(X_{s,u}^n(x))|^p dx \right) du \\
 &= \int_s^t \mathbb{E} \left( \int_{X_{s,u}^n(B(R))} |\xi_0^n(y) - \xi_0(y)|^p K_{s,u}^{n,-1}(y) dy \right) du,
 \end{aligned}$$



where

$$\begin{aligned} K_{s,u}^{n,-1}(y) &= \frac{d\lambda_{s,u}^{n,-1}}{d\lambda}(y) = \frac{d(\lambda \circ (X_{s,u}^n)^{-1})}{d\lambda}(y) \\ &= \exp\left(-\sum_{i=1}^N \int_s^u \xi_i((X_{r,u}^n)^{-1}(y)) \circ \hat{d}w_r^i - \int_s^u \xi_0^n((X_{r,u}^n)^{-1}(y)) dr\right), \end{aligned} \quad (3.9)$$

where  $\circ \hat{d}w_r^i$  denotes the backward Stratonovich stochastic integral. The estimate in Lemma 3.5 also holds for  $K_{s,u}^{n,-1}(y)$ . Let  $M > 0$  be big enough. We have

$$\begin{aligned} \int_{B(R)} \mathbb{E}(I_n^{3,1}) dx &\leq \int_s^t \mathbb{E}\left[\left(\int_{B(M)} + \int_{X_{s,u}^n(B(R)) \setminus B(M)}\right) |\xi_0^n(y) - \xi_0(y)|^p K_{s,u}^{n,-1}(y) dy\right] du \\ &= \int_s^t \int_{B(M)} |\xi_0^n(y) - \xi_0(y)|^p \mathbb{E}(K_{s,u}^{n,-1}(y)) dy du \\ &\quad + \int_s^t \mathbb{E}\left(\int_{X_{s,u}^n(B(R)) \setminus B(M)} |\xi_0^n(y) - \xi_0(y)|^p K_{s,u}^{n,-1}(y) dy\right) du. \end{aligned} \quad (3.10)$$

Since  $\xi_0^n$  and  $\xi_0$  are uniformly bounded, the second term on the right side of (3.10) is dominated by

$$\begin{aligned} C_p \int_s^t \mathbb{E}\left(\int_{X_{s,u}^n(B(R)) \setminus B(M)} K_{s,u}^{n,-1}(y) dy\right) du \\ = C_p \int_s^t \mathbb{E}[\lambda_{s,u}^{n,-1}(X_{s,u}^n(B(R)) \setminus B(M))] du \\ = C_p \int_s^t \mathbb{E}[\lambda(B(R) \setminus X_{s,u}^{n,-1}(B(M)))] du. \end{aligned} \quad (3.11)$$

Therefore we have by (3.10), (3.11) and Lemma 3.5,

$$\int_{B(R)} \mathbb{E}(I_n^{3,1}) dx \leq \tilde{C} \int_{B(M)} |\xi_0^n(y) - \xi_0(y)|^p dy + C_p \int_s^t \mathbb{E}[\lambda(B(R) \setminus X_{s,u}^{n,-1}(B(M)))] du.$$

From the definition of  $\xi_0^n$  we know that, for any  $p > 1$ ,  $\xi_0^n \rightarrow \xi_0$  in  $L_{loc}^p(\mathbb{R}^d)$ . By the Fatou lemma and the dominated convergence theorem, we obtain

$$\limsup_{n \rightarrow +\infty} \int_{B(R)} \mathbb{E}(I_n^{3,1}) dx \leq C_p \limsup_{n \rightarrow +\infty} \int_s^t \mathbb{E}[\lambda(B(R) \setminus X_{s,u}^{n,-1}(B(M)))] du$$

$$\begin{aligned}
&\leq C_p \int_s^t \mathbb{E} \left[ \limsup_{n \rightarrow +\infty} \lambda(B(R) \setminus X_{s,u}^{n,-1}(B(M))) \right] du \\
&\leq C_p \int_s^t \mathbb{E} [\lambda(B(R) \setminus X_{s,u}^{-1}(B(M-1)))] du \rightarrow 0
\end{aligned} \tag{3.12}$$

as  $M \rightarrow +\infty$ .

Now we handle with  $I_n^{3,2}$ . Since  $\xi_0$  is bounded, by the dominated convergence theorem, we only have to prove that for any  $u \in [s, 1]$ ,

$$\lim_{n \rightarrow +\infty} \mathbb{E} \int_{B(R)} |\xi_0(X_{s,u}^n(x)) - \xi_0(X_{s,u}(x))|^p dx = 0.$$

By the limit theorem (3.7), up to a subsequence, we have a.s.  $X_{s,t}^n(x)$  converges locally uniformly to  $X_{s,t}(x)$  on  $[s, 1] \times \mathbb{R}^d$ . For  $M > 0$ , define

$$\Omega_M = \left\{ w \in \Omega : \sup_{t \in [s, 1]} \sup_{x \in B(R)} |X_{s,t}(x)| \leq M \right\}.$$

Then (3.2) tells us that for any  $\varepsilon > 0$ , there is  $M$  big enough, such that  $\mathbb{P}(\Omega_M^c) < \varepsilon$ . We may assume at the same time that

$$\mathbb{E}[\lambda(B(R) \setminus X_{s,u}^{-1}(B(M)))] < \varepsilon. \tag{3.13}$$

By the Lusin theorem, there exists  $E_\varepsilon \subset B(M+1)$  so that  $\lambda(B(M+1) \setminus E_\varepsilon) < \varepsilon$  and the restriction  $\xi_0|_{E_\varepsilon}$  is uniformly continuous. We have

$$\mathbb{E} \left( \mathbf{1}_{\Omega_M^c} \int_{B(R)} |\xi_0(X_{s,u}^n(x)) - \xi_0(X_{s,u}(x))|^p dx \right) \leq (2\|\xi_0\|_\infty)^p \lambda(B(R)) \mathbb{P}(\Omega_M^c) < C_{p,R} \varepsilon.$$

Therefore

$$\begin{aligned}
&\limsup_{n \rightarrow +\infty} \mathbb{E} \int_{B(R)} |\xi_0(X_{s,u}^n(x)) - \xi_0(X_{s,u}(x))|^p dx \\
&\leq C_{p,R} \varepsilon + \limsup_{n \rightarrow +\infty} \mathbb{E} \left( \mathbf{1}_{\Omega_M} \int_{B(R)} |\xi_0(X_{s,u}^n(x)) - \xi_0(X_{s,u}(x))|^p dx \right).
\end{aligned} \tag{3.14}$$

For a.s.  $w \in \Omega_M$ , let

$$U_1^{(n)}(w) = \{x \in B(R) : X_{s,u}^n(x) \in E_\varepsilon, X_{s,u}(x) \in E_\varepsilon\},$$

$$U_2^{(n)}(w) = \{x \in B(R) : X_{s,u}^n(x) \in E_\varepsilon, X_{s,u}(x) \in E_\varepsilon^c\},$$

$$U_3^{(n)}(w) = \{x \in B(R) : X_{s,u}^n(x) \in E_\varepsilon^c, X_{s,u}(x) \in E_\varepsilon\},$$

$$U_4^{(n)}(w) = \{x \in B(R) : X_{s,u}^n(x) \in E_\varepsilon^c, X_{s,u}(x) \in E_\varepsilon^c\}.$$

Then

$$\begin{aligned}
& \mathbb{E} \left( \mathbf{1}_{\Omega_M} \int_{B(R)} |\xi_0(X_{s,u}^n(x)) - \xi_0(X_{s,u}(x))|^p dx \right) \\
&= \sum_{i=1}^4 \mathbb{E} \left( \mathbf{1}_{\Omega_M} \int_{U_i^{(n)}} |\xi_0(X_{s,u}^n(x)) - \xi_0(X_{s,u}(x))|^p dx \right) =: \sum_{i=1}^4 J_i^{(n)}. \quad (3.15)
\end{aligned}$$

Since  $\xi_0|_{E_\varepsilon}$  is uniformly continuous, there is  $\delta > 0$  such that for all  $y_1, y_2 \in E_\varepsilon$  and  $|y_1 - y_2| \leq \delta$ , we have  $|\xi_0(y_1) - \xi_0(y_2)|^p \leq \varepsilon$ . Thus from  $U_1^{(n)} \subset B(R)$  we obtain

$$\begin{aligned}
J_1^{(n)} &= \mathbb{E} \left[ \mathbf{1}_{\Omega_M} \left( \int_{U_1^{(n)} \cap \{|X_{s,u}^n - X_{s,u}| \leq \delta\}} + \int_{U_1^{(n)} \cap \{|X_{s,u}^n - X_{s,u}| > \delta\}} \right) \right. \\
&\quad \left. \times |\xi_0(X_{s,u}^n(x)) - \xi_0(X_{s,u}(x))|^p dx \right] \\
&\leq \varepsilon \lambda(B(R)) + \mathbb{E} \left( \mathbf{1}_{\Omega_M} \int_{\{x \in B(R): |X_{s,u}^n(x) - X_{s,u}(x)| > \delta\}} (2\|\xi_0\|_\infty)^p dx \right) \\
&= \varepsilon \lambda(B(R)) + (2\|\xi_0\|_\infty)^p \mathbb{E}[\lambda(\{x \in B(R): |X_{s,u}^n(x) - X_{s,u}(x)| > \delta\})].
\end{aligned}$$

Up to a subsequence, for a.s.  $w$ ,  $X_{s,u}^n$  converges uniformly to  $X_{s,u}$  on  $B(R)$ , therefore

$$\limsup_{n \rightarrow +\infty} J_1^{(n)} \leq \varepsilon \lambda(B(R)).$$

In the following, we write  $\tilde{C}_p = (2\|\xi_0\|_\infty)^p$ . By the definition of  $\Omega_M$  and Corollary 3.4,

$$\begin{aligned}
J_2^{(n)} &\leq \tilde{C}_p \mathbb{E} \left( \mathbf{1}_{\Omega_M} \int_{\{x \in B(R): X_{s,u}(x) \in E_\varepsilon^c\}} dx \right) \\
&= \tilde{C}_p \mathbb{E}[\mathbf{1}_{\Omega_M} \lambda(\{x \in B(R): X_{s,u}(x) \in E_\varepsilon^c \cap B(M+1)\})] \\
&\leq \tilde{C}_p \mathbb{E}[\mathbf{1}_{\Omega_M} \lambda(X_{s,u}^{-1}(B(M+1) \setminus E_\varepsilon))] \leq \tilde{C}_p C_1 \lambda(B(M+1) \setminus E_\varepsilon) < \tilde{C}_p C_1 \varepsilon.
\end{aligned}$$

The term  $J_3^{(n)}$  can be estimated as below:

$$\begin{aligned}
J_3^{(n)} &\leq \tilde{C}_p \mathbb{E} \left( \mathbf{1}_{\Omega_M} \int_{\{x \in B(R): X_{s,u}^n(x) \in E_\varepsilon^c\}} dx \right) = \tilde{C}_p \mathbb{E}[\mathbf{1}_{\Omega_M} \lambda(B(R) \cap X_{s,u}^{n,-1}(E_\varepsilon^c))] \\
&= \tilde{C}_p \mathbb{E}[\mathbf{1}_{\Omega_M} \lambda(B(R) \cap X_{s,u}^{n,-1}([E_\varepsilon^c \cap B(M+1)] \cup [E_\varepsilon^c \cap (B(M+1))^c]))] \\
&\leq \tilde{C}_p \mathbb{E}[\lambda(X_{s,u}^{n,-1}(B(M+1) \setminus E_\varepsilon))] + \tilde{C}_p \mathbb{E}[\mathbf{1}_{\Omega_M} \lambda(B(R) \cap X_{s,u}^{n,-1}((B(M+1))^c))] \\
&\leq \tilde{C}_p C_1 \lambda(B(M+1) \setminus E_\varepsilon) + \tilde{C}_p \mathbb{E}[\mathbf{1}_{\Omega_M} \lambda(B(R) \setminus X_{s,u}^{n,-1}(B(M+1)))] \\
&< \tilde{C}_p C_1 \varepsilon + \tilde{C}_p \mathbb{E}[\mathbf{1}_{\Omega_M} \lambda(B(R) \setminus X_{s,u}^{n,-1}(B(M+1)))],
\end{aligned}$$

where for the third inequality we have applied Corollary 3.4 to  $\lambda_{s,u}^{n,-1} = \lambda \circ X_{s,u}^{n,-1}$  (it is obvious that the results of Proposition 3.3 and Corollary 3.4 hold for  $\lambda_{s,t}^{n,-1}$  with the same constant  $C_p$  independent of  $n$ ). Hence by (3.13),

$$\begin{aligned} \limsup_{n \rightarrow +\infty} J_3^{(n)} &\leq \tilde{C}_p C_1 \varepsilon + \limsup_{n \rightarrow +\infty} \tilde{C}_p \mathbb{E}[\mathbf{1}_{\Omega_M} \lambda(B(R) \setminus X_{s,u}^{n,-1}(B(M+1)))] \\ &\leq \tilde{C}_p C_1 \varepsilon + \tilde{C}_p \mathbb{E}[\lambda(B(R) \setminus X_{s,u}^{-1}(B(M)))] \leq \tilde{C}_p C_1 \varepsilon + \tilde{C}_p \varepsilon, \end{aligned}$$

where  $\tilde{C}_p = (2\|\xi_0\|_\infty)^p$ . In the same way we obtain the estimate for  $J_4^{(n)}$ . Summing up the results for  $J_i^{(n)}$  ( $i = 1, \dots, 4$ ) and by (3.14), (3.15), we get

$$\limsup_{n \rightarrow +\infty} \mathbb{E} \int_{B(R)} |\xi_0(X_{s,u}^n(x)) - \xi_0(X_{s,u}(x))|^p dx \leq C'_{p,R} \varepsilon.$$

Since  $\varepsilon > 0$  is arbitrary, we arrive at the desired result and we conclude the proof of this proposition.  $\square$

Now we are in the position to prove Theorem 1.1.

**Proof of Theorem 1.1.** Denote by  $K_{s,t}(x)$  the right side of (1.5). By the proof of Lemma 3.5, it is clear that for any  $p > 1$ ,

$$\sup_{t \in [s, 1]} \sup_{x \in \mathbb{R}^d} \mathbb{E}((K_{s,t}(x))^p) < +\infty. \quad (3.16)$$

Fix any  $f \in C_c(\mathbb{R}^d)$ . There exists  $R > 0$  such that  $\text{supp}(f) \subset B(R)$ . By (3.7), a standard argument of reversing the time (e.g. see the proof of Theorem 5.1 in [8]) implies that for all  $t \in [s, 1]$ , up to a subsequence, we have a.s.,

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^d} f(X_{s,t}^{n,-1}(x)) dx = \int_{\mathbb{R}^d} f(X_{s,t}^{-1}(x)) dx = \int_{\mathbb{R}^d} f(y) d\lambda_{s,t}(y). \quad (3.17)$$

On the other hand, by (3.8),

$$\int_{\mathbb{R}^d} f(X_{s,t}^{n,-1}(x)) dx = \int_{\mathbb{R}^d} f(y) d\lambda_{s,t}^n(y) = \int_{\mathbb{R}^d} f(y) K_{s,t}^n(y) dy.$$

We have

$$\begin{aligned} K_{s,t}^n(y) - K_{s,t}(y) &= e^{\zeta_{s,t}^n(y)} - e^{\zeta_{s,t}(y)} = \int_0^1 \frac{d}{dr} (e^{\zeta_{s,t}(y) + r(\zeta_{s,t}^n(y) - \zeta_{s,t}(y))}) dr \\ &= (\zeta_{s,t}^n(y) - \zeta_{s,t}(y)) \int_0^1 e^{\zeta_{s,t}(y) + r(\zeta_{s,t}^n(y) - \zeta_{s,t}(y))} dr. \end{aligned}$$

Hence by the Cauchy inequality,

$$\left| \int_{\mathbb{R}^d} f(y) K_{s,t}^n(y) dy - \int_{\mathbb{R}^d} f(y) K_{s,t}(y) dy \right|^2$$

$$\begin{aligned} &\leq \|f\|_\infty^2 \left( \int_{B(R)} |K_{s,t}^n(y) - K_{s,t}(y)| dy \right)^2 \\ &\leq \|f\|_\infty^2 \left( \int_{B(R)} |\zeta_{s,t}^n(y) - \zeta_{s,t}(y)|^2 dy \right) \left( \int_{B(R)} \int_0^1 e^{2\zeta_{s,t}(y) + 2r(\zeta_{s,t}^n(y) - \zeta_{s,t}(y))} dr dy \right). \end{aligned}$$

Applying the Cauchy inequality again and by Lemma 3.5, (3.16),

$$\begin{aligned} &\mathbb{E} \left( \left| \int_{\mathbb{R}^d} f(y) K_{s,t}^n(y) dy - \int_{\mathbb{R}^d} f(y) K_{s,t}(y) dy \right|^2 \right) \\ &\leq C_{f,R} \left( \int_{B(R)} \mathbb{E}(|\zeta_{s,t}^n(y) - \zeta_{s,t}(y)|^4) dy \right)^{\frac{1}{2}} \\ &\quad \times \left( \int_0^1 \int_{B(R)} \mathbb{E}(e^{4\zeta_{s,t}(y) + 4r(\zeta_{s,t}^n(y) - \zeta_{s,t}(y))}) dr dy \right)^{\frac{1}{2}} \\ &\leq C'_{f,R} \left( \int_{B(R)} \mathbb{E}(|\zeta_{s,t}^n(y) - \zeta_{s,t}(y)|^4) dy \right)^{\frac{1}{2}}, \end{aligned}$$

where  $C_{f,R}$ ,  $C'_{f,R}$  are constants depending on  $f$  and  $R$ . Then by Proposition 3.6 we deduce that up to a subsequence,

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left( \left| \int_{\mathbb{R}^d} f(y) K_{s,t}^n(y) dy - \int_{\mathbb{R}^d} f(y) K_{s,t}(y) dy \right|^2 \right) = 0.$$

This and (3.17) lead to a.s.,

$$\int_{\mathbb{R}^d} f(y) d\lambda_{s,t}(y) = \int_{\mathbb{R}^d} f(y) K_{s,t}(y) dy.$$

Since both sides of the above equality are continuous functions of  $t \in [s, 1]$ , we conclude that a.s.

$$\int_{\mathbb{R}^d} f(y) d\lambda_{s,t}(y) = \int_{\mathbb{R}^d} f(y) K_{s,t}(y) dy, \quad \text{for all } s \leq t \leq 1.$$

Take a countable dense subset  $\{f_n: n \geq 1\} \subset C_c(\mathbb{R}^d)$ . We can find a subset  $\Omega_0 \subset \Omega$  with  $\mathbb{P}(\Omega_0) = 1$ , such that for every  $w \in \Omega_0$ , for all  $t \in [s, 1]$  and  $n \geq 1$ ,

$$\int_{\mathbb{R}^d} f_n(y) d\lambda_{s,t}(y) = \int_{\mathbb{R}^d} f_n(y) K_{s,t}(y) dy.$$

This implies that a.s. for all  $t \in [s, 1]$ ,  $\lambda_{s,t} \ll \lambda$  and  $K_{s,t} = \frac{d\lambda_{s,t}}{d\lambda}$ .  $\square$

**Remark 3.7.** It is worth pointing out the difference between the proofs of Theorems 1.1 and 2.2. In the proof of Theorem 2.2, we first show that the Lebesgue measure is quasi-invariant under the flow generated by (1.1) (see also [4]), while for Theorem 1.1, the quasi-invariance is proved after we have obtained the expression of the Radon–Nikodym derivative.

### 3.2. Explicit solution to Fokker–Planck equation

In this subsection we would like to give a brief discussion on the Fokker–Planck (or forward Kolmogorov) equation corresponding to the SDE (1.2), that is

$$\frac{\partial \mu_{s,t}}{\partial t} + \sum_{i=1}^d \partial_i (b_i \mu_{s,t}) - \frac{1}{2} \sum_{i,j=1}^d \partial_{ij} (a_{ij} \mu_{s,t}) = 0, \quad t \geq s, \quad \mu_{s,s} = \mu_0, \quad (3.18)$$

where

$$b_i = A_0^i + \sum_{l=1}^N \langle \nabla A_l^i, A_l \rangle, \quad a_{ij} = \sum_{l=1}^N A_l^i A_l^j, \quad i, j = 1, \dots, d. \quad (3.19)$$

When the coefficients are not smooth, Figalli [11] studied the relationship between the well-posedness of the martingale problem of the Itô SDE and the existence and uniqueness of measure-valued solutions to the Fokker–Planck equation (see the classical reference book [21] for a complete exposition of this issue). The author also gives sufficient conditions for the existence and uniqueness of solutions to (3.18), and in turn, these results are applied to conclude that the Itô SDE has a unique martingale solution. LeBris and Lions [16] studied systematically the Fokker–Planck type equations with irregular coefficients, establishing some results for the existence and uniqueness of solutions in suitable spaces.

Define the second order differential operator

$$L = \frac{1}{2} \sum_{i,j=1}^d a_{ij} \partial_{ij} + \sum_{i=1}^d b_i \partial_i,$$

or written in the following well-known form

$$L = \frac{1}{2} \sum_{i=1}^N A_i^2 + A_0.$$

A measure-valued function  $\mu_{s,t}$  on  $[s, T]$  is called a solution to the Fokker–Planck equation (3.18), if for any  $\varphi \in C_c^\infty(\mathbb{R}^d)$ , the equality

$$\frac{d}{dt} \int_{\mathbb{R}^d} \varphi(x) d\mu_{s,t}(x) = \int_{\mathbb{R}^d} L\varphi(x) d\mu_{s,t}(x)$$

holds in the distribution sense on  $[s, T]$  and  $\mu_{s,t}$  is  $w^*$ -convergent to  $\mu_0$  as  $t \downarrow s$ . The above equation can simply be written as

$$\frac{\partial \mu_{s,t}}{\partial t} = L^* \mu_{s,t}, \quad t \geq s, \quad \mu_{s,s} = \mu_0, \quad (3.20)$$

where  $L^*$  is the formal adjoint operator of  $L$ . If  $\mu_{s,t}$  is absolutely continuous with respect to the Lebesgue measure with a density function  $u_{s,t}$ , then  $u_{s,t}$  is also called a solution to (3.18). Denote

by  $\mathcal{M}_+^f$  the space of positive Radon measures on  $\mathbb{R}^d$  with finite total mass. For  $\mu_0 \in \mathcal{M}_+^f$ , by the Itô formula, it is easy to check that the measure defined below

$$\int_{\mathbb{R}^d} \varphi(x) d\mu_{s,t}(x) = \int_{\mathbb{R}^d} \mathbb{E}[\varphi(X_{s,t}(x))] d\mu_0(x), \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^d) \quad (3.21)$$

is a solution of (3.18), where  $X_{s,t}(x)$  is the solution to the SDE (1.2).

Besides the existence and uniqueness of solutions to (3.18), another problem which attracts our attention is whether the solution  $\mu_{s,t}$  has a density with respect to the Lebesgue measure  $\lambda$ . The classical theory of partial differential equations tells us that if the differential operator  $L$  is uniformly elliptic, the answer is positive even in the case that the initial condition  $\mu_0$  is a Dirac mass. In 1967, Hörmander proved a well-known result, which asserts that the hypoellipticity of  $L$  also implies that  $\mu_{s,t}$  is absolutely continuous with respect to  $\lambda$ , cf. [12, Chapter 5]. The following theorem gives a sufficient condition which guarantees the uniqueness of Eq. (3.18) (or equivalently (3.20)), and we also show in a special case that the unique solution has a density with respect to the Lebesgue measure. Notice that we do not assume the operator  $L$  is uniformly elliptic or hypoelliptic.

**Theorem 3.8.** *Suppose that the vector fields  $A_i \in C_b^{3+\delta}$ ,  $i = 1, \dots, N$ , and  $A_0 \in C_b$  satisfies (1.3), then for any  $\mu_0 \in \mathcal{M}_+^f$ , the Fokker–Planck equation (3.20) has a unique finite non-negative measure-valued solution.*

*Moreover, if the initial datum  $\mu_0 \ll \lambda$  and the density  $u_0 = \frac{d\mu_0}{d\lambda} \in C^3(\mathbb{R}^d)$  is strictly positive, then the unique solution  $\mu_{s,t}$  to (3.18) is absolutely continuous with respect to  $\lambda$ , and the Radon–Nikodym derivative is given by*

$$\frac{d\mu_{s,t}}{d\lambda} = u_0 k_{s,t}^{\mu_0},$$

where  $k_{s,t}^{\mu_0} = \mathbb{E}(K_{s,t}^{\mu_0})$  and  $K_{s,t}^{\mu_0}(x) = \frac{d(\mu_0 \circ X_{s,t}^{-1})}{d\mu_0}(x)$ .

**Proof.** Under these conditions, the Itô SDE with diffusion coefficients  $A_1, \dots, A_N$  and drift coefficient  $b$  (which is equivalent to the Stratonovich SDE (1.2)) defines a stochastic flow of homeomorphisms on  $\mathbb{R}^d$ , see [8, Theorem 1]. Therefore the martingale problem for the operator  $L$  is obviously well posed. By Lemma 2.3 of [11], we immediately obtain the desired result.

Now we prove the second assertion. To this end, let  $\mu_0 \in \mathcal{M}_+^f$  be absolutely continuous with respect to  $\lambda$  and  $u_0 := \frac{d\mu_0}{d\lambda} \in C^3(\mathbb{R}^d)$  is strictly positive. Then as has been mentioned at the end of Section 1,  $\mu_0$  is quasi-invariant under the stochastic flow  $X_{s,t}$  generated by (1.2), and the Radon–Nikodym derivative  $K_{s,t}^{\mu_0}(x) = \frac{d(\mu_0 \circ X_{s,t}^{-1})}{d\mu_0}(x)$  has an explicit expression of the form similar to (1.5). We have for any  $\varphi \in C_c^\infty(\mathbb{R}^d)$ ,

$$\int_{\mathbb{R}^d} \varphi(X_{s,t}(x)) d\mu_0(x) = \int_{\mathbb{R}^d} \varphi(y) d(\mu_0 \circ X_{s,t}^{-1})(y) = \int_{\mathbb{R}^d} \varphi(y) K_{s,t}^{\mu_0}(y) d\mu_0(y).$$

Therefore by (3.21),

$$\int_{\mathbb{R}^d} \varphi(x) d\mu_{s,t}(x) = \mathbb{E} \int_{\mathbb{R}^d} \varphi(y) K_{s,t}^{\mu_0}(y) d\mu_0 = \int_{\mathbb{R}^d} \varphi(y) k_{s,t}^{\mu_0}(y) d\mu_0(y),$$

which means that  $\frac{d\mu_{s,t}}{d\mu_0} = k_{s,t}^{\mu_0}$ , and hence the Radon–Nikodym derivative with respect to the Lebesgue measure

$$\frac{d\mu_{s,t}}{d\lambda} = \frac{d\mu_{s,t}}{d\mu_0} \cdot \frac{d\mu_0}{d\lambda} = k_{s,t}^{\mu_0} u_0.$$

The proof is complete.  $\square$

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